A Second Integral of Motion in a Triaxial Galaxy

N. Caranicolas
Department of Astronomy, University of Thessaloniki, Thessaloniki, Greece

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We construct a formal second integral of motion in a bisymmetrical potential field describing the motion on the plane of symmetry of a triaxial galaxy. We also check the constancy of the second integral adding higher-order terms which are found using a computer program.

Since the first indication that elliptical galaxies may not be rotationally flattened [1] and the confirmation of this by Illingworth [2] the dynamics of elliptical galaxies have been an active field of research.

In order to describe the motion on the plane of symmetry of a triaxial galaxy we adopt the bisymmetrical potential

$$V = \tfrac{1}{2}(Px^2 + Qy^2) - \varepsilon(a x^4 + 2b x^2 y^2 + c y^4), \qquad (1)$$

where P, Q, a, b, c are positive constants and ε is a small parameter. It is also supposed that the ratio $P^{1/2}/Q^{1/2}$ is irrational. Similar potentials with cubic [3], [4] or quartic [5] terms have been used to describe the motion on the meridian plane of biaxial galaxies.

In the case of the potential (1) the Hamiltonian

$$H = \tfrac{1}{2}(X^2 + Y^2 + Px^2 + Qy^2) - \varepsilon(a x^4 + 2b x^2 y^2 + c y^4) = H_0 + \varepsilon H_1 = h, \qquad (2)$$

where $X = \mathrm{d}x/\mathrm{d}t$, $Y = \mathrm{d}y/\mathrm{d}t$ and h is the numerical value of the function H, is an exact integral of motion.

In applying the method of Contopoulos [6] we shall try to construct a formal second integral of motion of the form

$$I = I_0 + \varepsilon I_1 + \dots, \qquad (3)$$

where

$$I_0 = \tfrac{1}{2}(X^2 + Px^2) \qquad (4)$$

and

$$I_{i+1} = -\int (I_i, H_1)\,\mathrm{d}t + f_{i+1}(I_0, J_0) \qquad (5)$$
$$(i = 0, 1, 2, \dots).$$

In (5), (I_i, H_1) are Poisson brackets, $J_0 = H_0 - I_0$, t is an auxiliary variable and $f_{i+1}(I_0, J_0)$ is an arbitrary function. Equation (5) for $i = 0$ gives

$$I_1 = -4\int (a x^3 X + b x y^2 X)\,\mathrm{d}t + f_1(I_0, J_0). \qquad (6)$$

If we take into account the solutions of the "unperturbed" problem ($\varepsilon = 0$)

$$\begin{aligned} x &= (2I_0/P)^{1/2} \sin P^{1/2}(t - t_0), \\ y &= (2J_0/Q)^{1/2} \sin Q^{1/2} t, \\ X &= (2I_0)^{1/2} \cos P^{1/2}(t - t_0), \\ Y &= (2J_0)^{1/2} \cos Q^{1/2} t, \end{aligned} \qquad (7)$$

where t_0 is a constant, we find

$$I_1 = \frac{bJ_0}{PQ}(X^2 - Px^2) - \frac{b}{P-Q}\left[\frac{1}{2Q}(X^2 - Px^2)(Y^2 - Qy^2) + 2xyXY\right] - a x^4 + f_1(I_0, J_0), \qquad (8)$$

where after finishing the integration we used the following identities:

$$\begin{aligned} 2I_0 \cos 2P^{1/2}(t - t_0) &= X^2 - Px^2, \\ I_0 \sin 2P^{1/2}(t - t_0) &= P^{1/2} x X, \\ 2J_0 \cos 2Q^{1/2} t &= Y^2 - Qy^2, \\ J_0 \sin 2Q^{1/2} t &= Q^{1/2} y Y. \end{aligned} \qquad (9)$$

Assuming that $f_1(I_0, J_0)$ is equal to zero we have the following expression for the first two terms of the second integral:

$$I = I_0 + \varepsilon I_1 = \tfrac{1}{2}(X^2 + Px^2) + \varepsilon\left\{\frac{bJ_0}{PQ}(X^2 - Px^2) - \frac{b}{P-Q}\left[\frac{1}{2Q}(X^2 - Px^2) \cdot (Y^2 - Qy^2) + 2xyXY\right] - a_1 x^4\right\}. \qquad (10)$$

The higher order terms of the second integral are given through the recurrent formula (5). However,

Reprint requests to Dr. N. Caranicolas, Department of Astronomy, University of Thessaloniki, Thessaloniki, Greece.

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Table 1. Constancy of the second integral.

I (Degree)	max	min	D	$2D/(\max+\min)$
$I_0(2)$	0.0100	0.0074	0.0026	0.30
$+\varepsilon I_1(4)$	0.00862	0.00850	0.00012	0.014
$+\varepsilon^2 I_2(6)$	0.008611	0.008594	0.000017	0.0020
$+\varepsilon^3 I_3(8)$	0.0086002	0.0085984	0.0000018	0.00021
$+\varepsilon^4 I_4(10)$	0.00859982	0.00859967	0.00000015	0.000017
$+\varepsilon^5 I_5(12)$	0.00859978	0.00859977	0.00000001	0.0000012
$+\varepsilon^6 I_6(14)$	0.00859977	0.00859977	0.00000000	0.00000000

the actual calculation of these terms becomes soon prohibitively long.

Using a computer program [7], we calculated the higher order terms of the second integral up to the terms of the fourteenth degree in the variables. The number of terms of different degrees are (2) 2; (4) 8; (6) 20; (8) 40; (10) 70; (12) 112; (14) 168. It is interesting to note that as higher order terms are included in the expansion (10) the second integral is better conserved when the energy is small. For this reason the values of the second integral were found for a number of orbits in many points along the orbit. The orbits were calculated by numerical integration of the equations of motion using the Runge-Kutta method in double precision. The units used were kpc for x, y and $kpc\,(10^7\,yr)^{-1}$ for X, Y. The step of integration was $0.02\,(10^7\,yr)$.

In Table 1 max and min are the maximum and minimum values of the second integral for an orbit with initial conditions $x_0 = 0.2$, $y_0 = X_0 = 0$ and $h = 0.1$, for a time interval equal to 5×10^9 years. The values of the constants are $P = 0.5$, $Q = 0.6$, $a = 1.0$, $b = 0.8$, $c = 0.5$ while ε is equal to 0.05. The maximum deviation $D = \max - \min$ and the maximum relative deviation $2D/(\max + \min)$ are also given in this Table. We see that when the terms of the fourteenth degree are included 8 significant figures of the second integral are constant. The energy constant was conserved up to the twelveth printed figure.

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